Modeling and simulation of contact between rigid bodies

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Stokes equations :

$$\begin{cases} -\mu\Delta u + \nabla p &= f,\ \Omega, \\ \nabla \cdot u &= 0,\ \Omega. \end{cases}$$

Fluid definitions :

- $\Omega \subseteq \mathbb{R}^d$, fluid domain.
- $u : \Omega \to \mathbb{R}^d$, velocity.
- $p : \Omega \to \mathbb{R}$, pressure.
- $\mu \in \mathbb{R}^+$, viscosity.
- $f : \Omega \to \mathbb{R}^d$, external forces, $f = 0_{\mathbb{R}^d}$.

2D fluid-spherical body problem :

$$\begin{cases} -\mu\Delta u + \nabla p & = & f,\, \Omega, \\ \nabla \cdot u & = & 0,\, \Omega, \\ u & = & U + \omega \times (x - x^{CM}),\, \partial P, \\ m\frac{dU}{dt} & = & F - \int_{\partial P} \sigma \cdot \overrightarrow{n}, \\ J\frac{d\omega}{dt} & = & -\int_{\partial P} \sigma \cdot \overrightarrow{n} \times (x - x^{CM}), \end{cases}$$

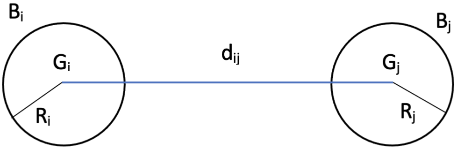
where F represents a repulsive force to model the contact between spherical bodies.

Fluid definitions :
- $\Omega \subseteq \mathbb{R}^2$, domain.
- $u : \Omega \to \mathbb{R}^2$, velocity.
- $p : \Omega \to \mathbb{R}$, pressure.
- $\mu \in \mathbb{R}^+$, viscosity.
- $\sigma = -pI_2 + \mu(\nabla u + \nabla u^T)$, total stress tensor. - $f : \Omega \to \mathbb{R}^2$, external forces. $f = 0_{\mathbb{R}^2}$.

Structure P definitions :
- $x^{CM} \in \mathbb{R}^2$, center of mass .
- $m \in \mathbb{R}^+$, mass.
- $J \in \mathbb{R}$, moment of inertia.
- $U \in \mathbb{R}^2$, translational velocity.
- $\omega \in \mathbb{R}$, angular velocity.

Collisions between circular rigid bodies

Smooth collision : The velocities of the rigid bodies coincide at the points of contact.



The repulsion force $\overrightarrow{F_{ij}}$ has to verify three properties :

- $\overrightarrow{F_{ij}}$ is parallel to $\overrightarrow{G_i G_j}$.
- $|\overrightarrow{F_{ij}}| = 0 \quad$ if $d_{ij} - R_i - R_j \geq \rho$,
  where ρ the range of the repulsion force.

R. Glowinski, T. W. Pan, T. I. Hesla, D. D. Joseph, and J. Périaux. *A Fictitious Domain Approach to the Direct Numerical Simulation of Incompressible Viscous Flow past Moving Rigid Bodies : Application to Particulate Flow*, 2000.

- For $0 \leq d_{ij} - R_i - R_j \leq \rho$:

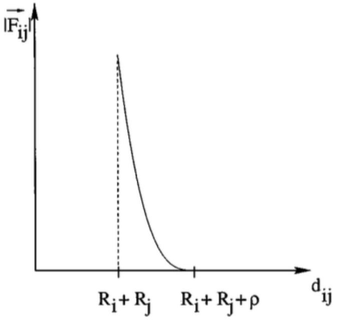


Figure – Repulsion force [1]

For $d_{ij} - R_i - R_j = 0$: $|\overrightarrow{F_{ij}}| = \frac{c_{ij}}{\epsilon}$,
where c_{ij} a scaling factor and $\epsilon > 0$ the collision parameter.

Definition of $\overrightarrow{F_{ij}}$:

$$\overrightarrow{F_{ij}} = \frac{c_{ij}}{\epsilon}(\max\{0, -(\frac{d_{ij} - R_i - R_j - \rho}{\rho})\})^2 \frac{\overrightarrow{G_i G_j}}{d_{ij}},$$

where $(\max\{0, -(\frac{d_{ij}-R_i-R_j-\rho}{\rho})\})^2$ a quadratic activation term.

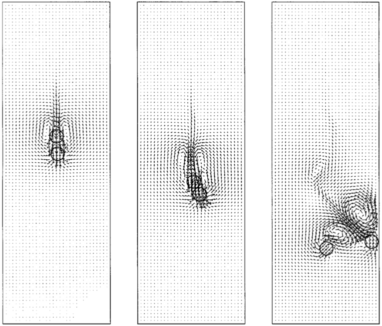


Figure – *Drafting, kissing and tumbling* phenomenon [1]

Stokes equations :

$$\begin{cases} -\mu\Delta u + \nabla p &= f \,, \quad \text{in } \Omega, \\ \nabla \cdot u &= 0 \,, \quad \text{in } \Omega, \\ u(x) &= h(x) \,, \quad \text{on } \Gamma_D, \\ \sigma(x)\vec{n} &= (-pI_d + 2\mu D(u))\vec{n} = g(x) \,, \quad \text{on } \Gamma_N. \end{cases}$$

where $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$.

Variational formulation :
Find $(u, p) \in X = H^1(\Omega)^d \times M = L^2_0(\Omega)$ such that :

$$2\mu \int_\Omega D(u) : D(v) - \int_\Omega p\nabla \cdot v = \int_{\Gamma_N} g(x) \cdot v + \int_\Omega f \cdot v \,, \quad \forall v \in X,$$

$$\int_\Omega q\nabla \cdot u = 0 \,, \quad \forall q \in M.$$

Notations : $a(u,v) = 2\mu \int_\Omega D(u) : D(v)$,$b(v,p) = -\int_\Omega p \nabla \cdot v$,
$F(v) = \int_\Omega f \cdot v$,$G(v) = \int_{\Gamma_N} g(x) \cdot v$.

Discrete variational formulation : Find $(u_h, p_h) \in X_h \subset X \times M_h \subset M$ such that :

$$a(u_h, v_h) + b(v_h, p_h) = G(v_h) + F(v_h) \ , \quad \forall v_h \in X_h,$$
$$b(u_h, q_h) = 0 \ , \quad \forall q_h \in M_h.$$

Matrix formulation :

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F + G \\ 0 \end{bmatrix},$$

where $A = (a(\phi_i, \phi_j))_{i,j}$, $\quad B = (b(\phi_i, \psi_j))_{i,j}$,
$U = (u_i)_i$, $\quad P = (p_i)_i$, $\quad F = (F(\phi_j))_j$, $\quad G = (G(\phi_j))_j$.

Non-homogenous Dirichlet boundary conditions 2D :

Exact solutions :

$$u = \begin{bmatrix} x^3 + x^2 + x^2y + x - 2xy - 3xy^2 \\ -3x^2y - 2xy - xy^2 - y + y^2 + y^3 \end{bmatrix},$$

$$p = x^3y^2 + x + xy + y - \frac{4}{3}.$$

h	$\|u - u_h\|_{L^2}$	Order	$\|p - p_h\|_{L^2} + \|u - u_h\|_{H^1}$	Order
$\frac{1}{4}$	0.0009034		0.0410482	
$\frac{1}{8}$	0.0001100	3.03	0.0099241	2.04
$\frac{1}{16}$	$1.41722e-05$	2.95	0.0025055	1.98
$\frac{1}{32}$	$1.7553e-06$	3.01	0.0006244	2.00
$\frac{1}{64}$	$2.17241e-07$	3.01	0.0001549	2.01
$\frac{1}{128}$	$2.71264e-08$	3.00	$3.86969e-05$	2.00

Non-homogenous Dirichlet boundary conditions 3D :
Exact solutions :

$$u = \begin{bmatrix} x^3y + x^2 + x + xy \\ x^2y^2 + xy + y + y^2 \\ -5 * x^2yz - 3xz - 3yz - 2z \end{bmatrix},$$

$$p = x^3y^3z + xyz - \frac{5}{32}.$$

h	$\|u - u_h\|_{L^2}$	Order	$\|p - p_h\|_{L^2} + \|u - u_h\|_{H^1}$	Order
1	0.0242545		0.7187319	
$\frac{1}{2}$	0.00615454	1.97	0.289773	1.31
$\frac{1}{4}$	0.00197867	1.63	0.1073367	1.43
$\frac{1}{8}$	0.000264588	2.90	0.02356425	2.18
$\frac{1}{16}$	$3.45405e-05$	2.93	0.00551451	2.09

Fluid-structure simulations

Falling confined disk :

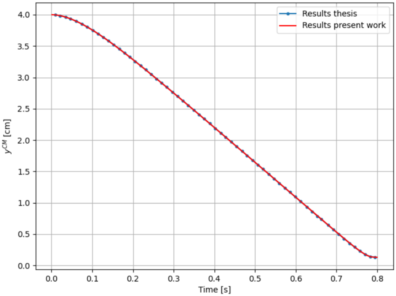


Figure – Time evolution of the y coordinate of the center of mass

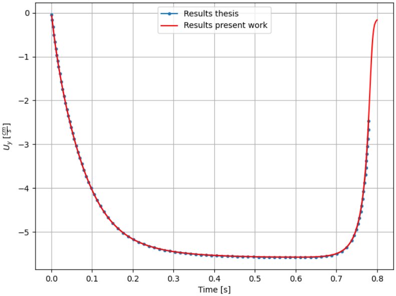
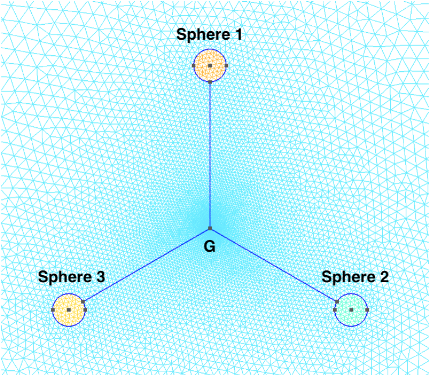


Figure – Time evolution of the vertical velocity

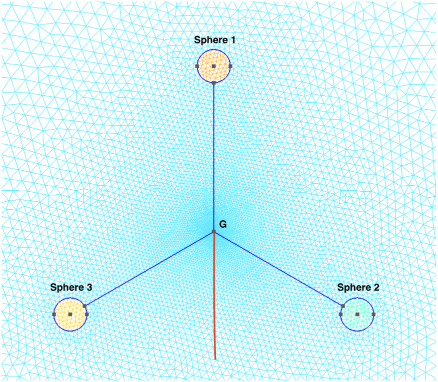
Three spheres planar swimmer :



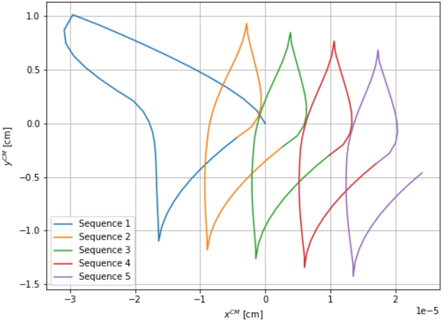
Sequence of movements :

1. Contraction of a sphere.
2. Contraction of the other two spheres.
3. Extension of the first sphere.
4. Extension of the other two spheres.

Theoretical displacement :



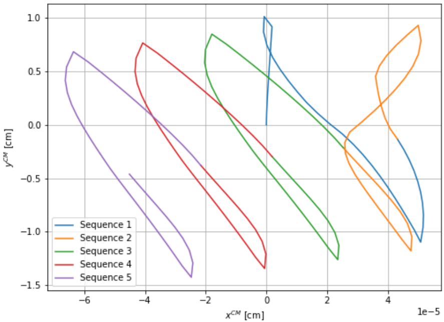
Observed displacement :



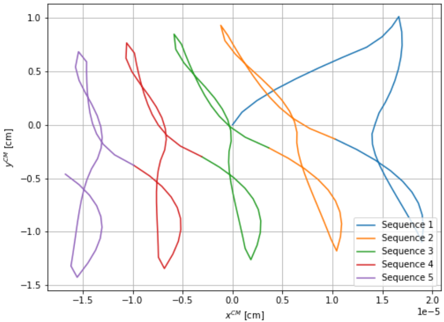
Number of remeshes : 2.
The moments of remeshing : $[0.9s, 1.7s]$.

Fluid-structure simulations

Displacement for sphere 2 :



Displacement for sphere 3 :



Perspectives

Collision theory :

- Analysis of the influence of the parameters used to define the repulsion force.
- Read other articles on collision treatment.

C++ implementation :

- Implementation of fluid-structure problem.
- Implementation of the unsteady Stokes equations.

Simulations with the fluid toolbox

- Add the repulsive force term in the script.
- Perform other tests with the three spheres planar swimmer.

References

R. Glowinski, T. W. Pan, T. I. Hesla, D. D. Joseph, and J. Périaux. *A Fictitious Domain Approach to the Direct Numerical Simulation of Incompressible Viscous Flow past Moving Rigid Bodies : Application to Particulate Flow*, 2000.

Laetitia Giraldi, Vincent Chabannes, Christophe Prud'homme, Luca Berti. *Benchmarking rigid bodies moving in fluids using Feel++*, 2022.

Luca Berti. *Numerical methods and optimisation for micro-swimming*, 2021.

Fluid-structure problem :

$$\left\{\begin{array}{rcl} -\mu\Delta u + \nabla p & = & f,\, \Omega\,, \\ \nabla\cdot u & = & 0,\, \Omega\,, \\ u & = & U + \omega \times (x - x^{CM}),\, \partial P\,, \\ m\frac{dU}{dt} & = & F - \int_{\partial P} \sigma \cdot \overrightarrow{n}\,, \\ J\frac{d\omega}{dt} & = & M - \int_{\partial P} \sigma \cdot \overrightarrow{n} \times (x - x^{CM})\,. \end{array}\right.$$

Variational formulation : Find $(u, p, U, \omega) \in H^1(\Omega)^d \times L^2_0(\Omega) \times T = \{U | U(t) \in \mathbb{R}^d\} \times W = \{\omega | \omega(t) \in \mathbb{R}^{d^*}\}$ such that :

$$2\mu \int_\Omega D(u) : D(v) - \int_\Omega p\nabla\cdot v = \int_{\partial P} \sigma \cdot \overrightarrow{n} \cdot v + \int_\Omega f \cdot v\ , \quad \forall v \in H^1(\Omega)^d,$$

$$\int_\Omega q\nabla\cdot u = 0\ , \quad \forall q \in L^2_0(\Omega),$$

$$m\frac{dU}{dt} \cdot \tilde{U} = F \cdot \tilde{U} - \int_{\partial P} \sigma \cdot \overrightarrow{n} \cdot \tilde{U}\ , \quad \forall \tilde{U} \in T,$$

$$J\frac{d\omega}{dt} \cdot \tilde{\omega} = M \cdot \tilde{\omega} - \int_{\partial P} \sigma \cdot \overrightarrow{n} \times (x - x^{CM}) \cdot \tilde{\omega}\ , \quad \forall \tilde{\omega} \in W.$$

Conditions on ∂P :

$$
\begin{aligned}
v &= \tilde{U} + \tilde{\omega} \times (x - x^{CM}) \\
\Rightarrow \int_{\partial P} \sigma \cdot \overrightarrow{n} \cdot v &= \int_{\partial P} \sigma \cdot \overrightarrow{n} \cdot \tilde{U} + \int_{\partial P} \sigma \cdot \overrightarrow{n} \times (x - x^{CM}) \cdot \tilde{\omega} \\
\Rightarrow \int_{\partial P} \sigma \cdot \overrightarrow{n} \cdot v &= -m \frac{dU}{dt} \cdot \tilde{U} + F \cdot \tilde{U} - J \frac{d\omega}{dt} \cdot \tilde{\omega} + M \cdot \tilde{\omega}.
\end{aligned}
$$

Final Variational formulation : Find $(u, p, U, \omega) \in H^1(\Omega)^d \times L^2_0(\Omega) \times T = \{U | U(t) \in \mathbb{R}^d\} \times W = \{\omega | \omega(t) \in \mathbb{R}^{d^*}\}$ such that :

$$
\begin{aligned}
2\mu \int_\Omega D(u) : D(v) - \int_\Omega p \nabla \cdot v &= -m \frac{dU}{dt} \cdot \tilde{U} + F \cdot \tilde{U} - J \frac{d\omega}{dt} \cdot \tilde{\omega} + M \cdot \tilde{\omega} + \int_\Omega f \cdot v, \\
\int_\Omega q \nabla \cdot u &= 0.
\end{aligned}
$$

where $v \in H^1(\Omega)^d, \tilde{U} \in T, \tilde{\omega} \in W, q \in L^2_0(\Omega)$.

Appendix : Distance function

Required : Function to determine the distance between the boundaries of the bodies.

Problem : Function executed in sequential and thus expensive.

Solution :

- The repulsive force only applied on a small area.
- Set a limit to which the distance is computed.