Modeling and simulation of contact between rigid bodies

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Fluid model with moving rigid bodies

Navier-Stokes equations :

$$\begin{cases} \rho_f \frac{du}{dt} + \rho_f (u \cdot \nabla) u - \nabla \cdot \sigma & = & 0_{\mathbb{R}^d}, \Omega, \\ \nabla \cdot u & = & 0, \quad \Omega, \end{cases}$$

where σ the total stress tensor :

$$\sigma = -pI_d + \mu[\nabla u + (\nabla u)^T].$$

Fluid definitions :

- $d = 2$ or $d = 3$, dimension.
- $\Omega \subseteq \mathbb{R}^d$, fluid domain.
- $u : \Omega \to \mathbb{R}^d$, velocity.
- $p : \Omega \to \mathbb{R}$, pressure.
- $\mu \in \mathbb{R}^+$, viscosity.
- $\rho_f \in \mathbb{R}^+$, density.
- $I_d \in \mathbb{M}_{d,d}$, identity tensor.

Fluid model with moving rigid bodies

The bodies can translate and rotate with gravity, fluid forces acting on them and collision forces in body-body and body-wall interactions.

Newton equations :

$$m_i \frac{dU_i}{dt} = (m_i - \int_{B_i} \rho_f) g + F_i + F_i^{'},$$

$$\frac{d[R J_i R^T \omega_i]}{dt} = T_i + T_i^{'},$$

***i*-th body definitions :**

- $B_i \subseteq \mathbb{R}^d$, body domain.
- $m_i \in \mathbb{R}$, mass.
- $J_i \in \mathbb{R}$ if $d = 2$, else $J_i \in \mathbb{R}^d$, moment of inertia.
- $R \in \mathbb{M}_{d,d}$, rotation matrix.
- $U_i \in \mathbb{R}^d$, translational velocity.
- $\omega_i \in \mathbb{R}$ if $d = 2$, else $\omega_i \in \mathbb{R}^d$, angular velocity.
- $x_i^{CM} \in \mathbb{R}^d$, center of mass.

where $g \in \mathbb{R}^d$ is the gravity vector, $F_i^{'} \in \mathbb{R}^d$, $T_i^{'} \in \mathbb{R}^d$ the collision forces and torque, and $F_i \in \mathbb{R}^d$, $T_i \in \mathbb{R}^d$ the hydrodynamic forces and torque acting on the body :

$$F_i = -\int_{\partial B_i} \sigma \cdot \overrightarrow{n} \quad \text{and} \quad T_i = -\int_{\partial B_i} \sigma \cdot \overrightarrow{n} \times (x_i - x_i^{CM}).$$

Fluid model with moving rigid bodies

Coupling condition on the velocities on ∂B_i :

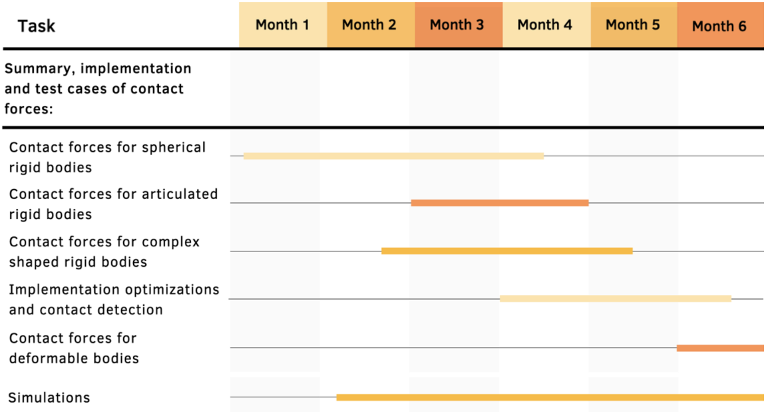
$$u = U_i + \omega_i \times (x - x_i^{CM}).$$

Fluid-body problem :

$$\begin{cases} \rho_f \frac{du}{dt} + \rho_f (u \cdot \nabla) u - \nabla \cdot \sigma &= 0_{\mathbb{R}^d}, \quad \text{in } \Omega, \\ \nabla \cdot u &= 0, \quad \text{in } \Omega, \\ u &= U_i + \omega_i \times (x - x_i^{CM}), \quad \text{on } \partial B_i, \\ m_i \frac{dU_i}{dt} &= (m_i - \int_{B_i} \rho_f) g + F_i + F_i^{'}, \\ \frac{d[R J_i R^T \omega_i]}{dt} &= T_i + T_i^{'}. \end{cases}$$

Decheng Wan, Stefan Turek. *Direct Numerical Simulation of Particulate Flow via Multigrid FEM Techniques and the Fictitious Boundary Method*, 2004.

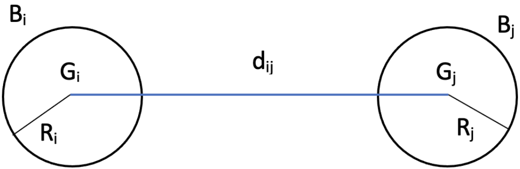
Gantt chart



Full Gantt chart : *https ://feelpp.github.io/swimmer/swimmer/latest/StageCeline/Introduction.html*.

Collisions between spherical rigid bodies

Smooth collision : The velocities of the rigid bodies coincide at the points of contact.



We note $d_{ij} = ||G_i - G_j||_2$, distance between the disk mass centers.

The repulsion force $\overrightarrow{F_{ij}}$ has to verify three properties :

1. $\overrightarrow{F_{ij}}$ is parallel to $\overrightarrow{G_i G_j}$.

R. Glowinski, T. W. Pan, T. I. Hesla, D. D. Joseph, and J. Périaux. *A Fictitious Domain Approach to the Direct Numerical Simulation of Incompressible Viscous Flow past Moving Rigid Bodies : Application to Particulate Flow*, 2000.

2. $|\overrightarrow{F_{ij}}| = 0$ if $d_{ij} > R_i + R_j + \rho$, where ρ the range of the repulsion force.
3. For $R_i + R_j \leq d_{ij} \leq R_i + R_j + \rho$:

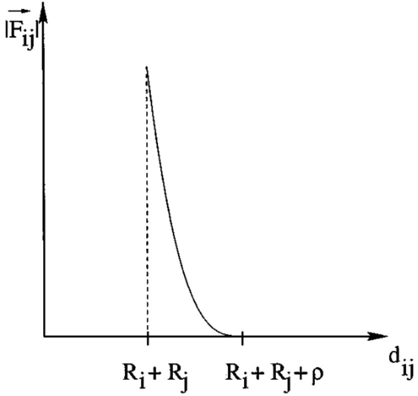


Figure – Repulsion force [1]

Collisions between spherical rigid bodies

Definition of $\overrightarrow{F_{ij}}$ for body-body interaction :

$$|\overrightarrow{F_{ij}}| = \begin{cases} 0\,, & \text{for} \quad d_{ij} > R_i + R_j + \rho, \\ \frac{1}{\epsilon}(G_i - G_j)(R_i + R_j + \rho - d_{ij})^2\,, & \text{for} \quad R_i + R_j \leq d_{ij} \leq R_i + R_j + \rho, \\ \frac{1}{\epsilon'}(G_i - G_j)(R_i + R_j - d_{ij})\,, & \text{for} \quad d_{ij} < R_i + R_j. \end{cases} .$$

where $(R_i + R_j + \rho - d_{ij})^2$ a quadratic activation term and $(G_i - G_j)$ gives the direction of the force.

The stiffness parameters are set to $\epsilon \approx h^2$ and $\epsilon' \approx h$, if :

- $\rho \approx h$, h the mesh step.
- $\frac{\rho_b}{\rho_f} \approx 1$, ρ_b the body density and ρ_f the fluid density.
- The fluid is sufficiently viscous.

Decheng Wan, Stefan Turek. *Direct Numerical Simulation of Particulate Flow via Multigrid FEM Techniques and the Fictitious Boundary Method*, 2004.

Collisions between spherical rigid bodies

Definition of $\overrightarrow{F_i^W}$ for body-wall interaction :

$$|\overrightarrow{F_i^W}| = \begin{cases} 0 \,, & \text{for} \quad d_i > 2R_i + \rho, \\ \frac{1}{\epsilon_W}(G_i - G_i^{'})(2R_i + \rho - d_i)^2 \,, & \text{for} \quad 2R_i \leq d_i \leq 2R_i + \rho, \\ \frac{1}{\epsilon_W^{'}}(G_i - G_i^{'})(2R_i - d_i) \,, & \text{for} \quad d_i < 2R_i. \end{cases} .$$

where $\rho \approx h$, $\epsilon_W = \frac{\epsilon}{2}$ and $\epsilon_W^{'} = \frac{\epsilon^{'}}{2}$.

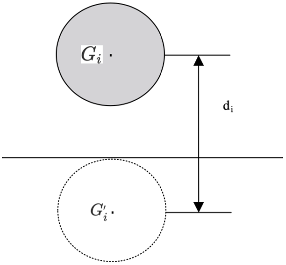


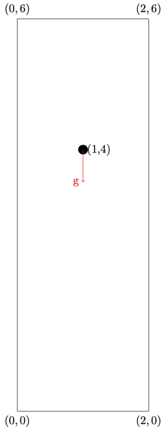
Figure – Imaginary body [2]

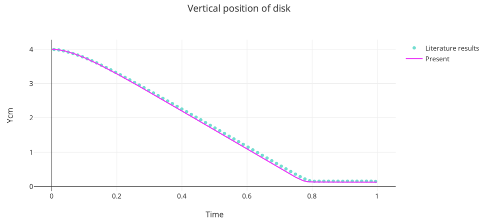
Total repulsion force applied on the body B_i :

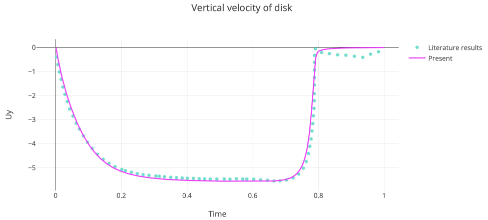
$$|\overrightarrow{F_i}| = \sum_{j=1, j\neq i}^{N} |\overrightarrow{F_{ij}}| + |\overrightarrow{F_i^W}|,$$

where N the number of bodies.

Simulation : Disk-bottom interaction



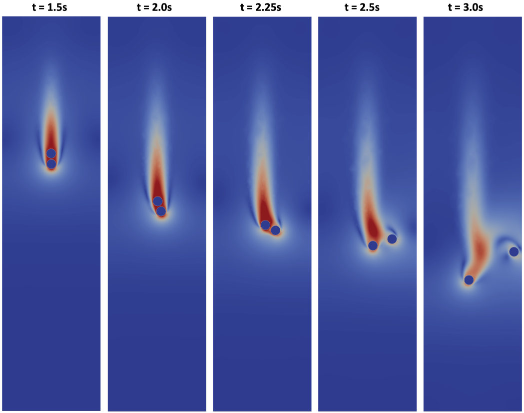




The differences in vertical velocity can be explained by a higher repulsion force intensity used for the literature results [3].

Simulation : Two disks interaction

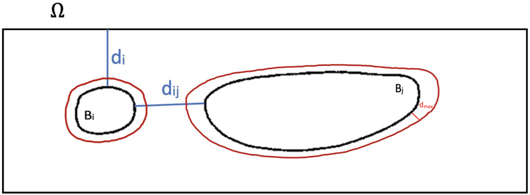
Drafting, kissing and tumbling phenomenon



K. Usman, K. Walayat, R. Mahmood, et al. *Analysis of solid particles falling down and interacting in a channel with sedimentation using fictitious boundary method*, 2018.

Decheng Wan, Stefan Turek. *Direct Numerical Simulation of Particulate Flow via Multigrid FEM Techniques and the Fictitious Boundary Method*, 2004.

Collisions between complex shaped rigid bodies



Narrow band fast marching method :

- Numerical technique for tracking the evolution of fronts propagating in direction normal to itself.
- The method computes the distance function $D_i(X) : \mathbb{R}^d \longrightarrow \mathbb{R}$ for each point of the mesh starting from a front ∂B_i :

$$\begin{cases} D_i(X) & = & d, \quad \text{where } d \text{ distance from } \partial B_i \text{ to point } X \in \mathbb{R}^d \\ D_i|_{\partial B_i} & = & 0. \end{cases}$$

- Narrow band : $D_i(X)$ only computed in a near neighborhood of thickness d_{max} around ∂B_i.

J.A. Sethian. *A fast marching level set method for monotonically advancing fronts*,1996.

Collisions between complex shaped rigid bodies

Reformulation of $\overrightarrow{F_{ij}}$ for body-body and $\overrightarrow{F_i^W}$ for body-wall interaction :

$$|\overrightarrow{F_{ij}}| = \begin{cases} 0\,, & \text{for} \quad d_{ij} > \rho, \\ \frac{1}{\epsilon}(\arg\min_{X\in\partial B_i} D_j(X) - \arg\min_{X\in\partial B_j} D_i(X))(\rho - d_{ij})^2\,, & \text{for} \quad 0 \leq d_{ij} \leq \rho. \end{cases}$$

$$|\overrightarrow{F_i^W}| = \begin{cases} 0\,, & \text{for} \quad d_i > \rho, \\ \frac{1}{\epsilon_W}(\arg\min_{X\in\partial B_i} D_\Omega(X) - \arg\min_{X\in\partial\Omega} D_i(X))(\rho - d_i)^2\,, & \text{for} \quad 0 \leq d_i \leq \rho. \end{cases}$$

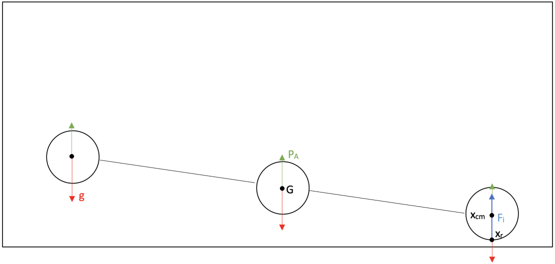
Angular momentum of $\overrightarrow{F_i} = \overrightarrow{F_{ij}} + \overrightarrow{F_i^W}$ for body-body and body-wall interaction :

$$\frac{d[RJ_iR^T\omega_i]}{dt} = T_i - \overrightarrow{Gx_r} \times \overrightarrow{F_i},$$

where G the mass center of the body and x_r the point where $\overrightarrow{F_i}$ applies on the body B_i.

Tsorng-Whay Pan, Roland Glowinski, Giovanni P.Galdi. *Direct simulation of the motion of a settling ellipsoid in Newtonian fluid*, 2001.

Aligned three-sphere swimmer :



Newton's equation describing the angular velocity :

$$\frac{d[RJ_iR^T\omega_i]}{dt} = T_i - \overrightarrow{Gx_r} \times \overrightarrow{F_i} + \overrightarrow{Gx_{cm}} \times \overrightarrow{(g - P_A)},$$

where G the mass center of the swimmer, x_r the point where $\overrightarrow{F_i}$ applies on the body, x_{cm} the mass center of one body, g the gravity vector and P_A the buoyancy.

Conclusions of midterm internship

Regarding implementation and algorithms :
Computational costs of the three algorithms were compared. The narrow band fast marching method proves to be efficient to determine the distance, nevertheless for 3D simulations optimizations will be necessary.

Regarding simulations and benchmarks :
Various simulations have been performed to validate the models. Even if the results obtained do not always correspond exactly to the benchmarks, the same behavior is found.

Regarding other results :
The implementation of the resolution of Stokes equations has been validated with benchmarks. The simulations performed with the three-sphere planar swimmer allow to conclude that its displacement will be the closest to the theoretical one when it is placed in a circular box of much larger size.

Internship perspectives

Simulation perspectives :

- Multi-body simulations.
- Motion of a particle placed in an artery or zebra fish.
- Motion of a particle placed in fluids with imposed velocity.
- Motion of three-sphere swimmer near wall.
- 3D simulations.

Theoretical perspectives :

- Considering literature on other approaches for contact force and detection.
- Considering simple cases of deformable bodies.

References

- R. Glowinski, T. W. Pan, T. I. Hesla, D. D. Joseph, and J. Périaux. *A Fictitious Domain Approach to the Direct Numerical Simulation of Incompressible Viscous Flow past Moving Rigid Bodies : Application to Particulate Flow*, 2000.
- P. Singh, T.I. Hesla, D.D. Joseph. *Distributed Lagrange multiplier method for particulate flows with collisions*, 2002.
- Decheng Wan, Stefan Turek. *Direct Numerical Simulation of Particulate Flow via Multigrid FEM Techniques and the Fictitious Boundary Method*, 2004.
- Laetitia Giraldi, Vincent Chabannes, Christophe Prud'homme, Luca Berti. *Benchmarking rigid bodies moving in fluids using Feel++*, 2022.

References

J.A. Sethian. *A fast marching level set method for monotonically advancing fronts*,1996.

J.A. Sethian. *Level Set Methods and Fast Marching Methods ; Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision and Materials Sciene*,1996.

Tsorng-Whay Pan, Roland Glowinski, Giovanni P.Galdi. *Direct simulation of the motion of a settling ellipsoid in Newtonian fluid*, 2001.